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ON THE STABILITY OF NONLINEAR OPERATOR DIFFERENTIAL EQUATIONS, AND APPLICATIONS

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ABSTRACT

Consider the nonlinear operator differential equation (i.e., equation of evolution)

$$(*) \qquad \frac{dx(t)}{dt} = Ax(t) + f(x(t)) \qquad (t \geqq 0)$$

where A is a linear (unbounded) operator with domain and range both in a real Hilbert space H and f is a (nonlinear) function defined on H into H. The object of this paper is to investigate the existence, the uniqueness and the stability or asymptotic stability of solutions to (*) by using nonlinear semi-group properties. Criteria on A and on f for the generation of a contraction or negative contraction semi-group are established from which the existence, uniqueness, stability and asymptotic stability of solutions of (*) are insured. Applications are given to the second order partial differential equation of the form

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right) + c(x)u + f(x,u) \qquad x \in \Omega < R^n.$$

Criteria in terms of the coefficients $a_{ij}(x)$, $c(x)$ and of the function f are obtained.

ON THE STABILITY OF NONLINEAR OPERATOR DIFFERENTIAL EQUATIONS, AND APPLICATIONS

1. Introduction

This paper is concerned with the existence and the stability problems of the operator differential equations (i.e., equations of evolution) of the form

$$\frac{dx(t)}{dt} = A\,x(t) + f(x(t)) \qquad t \geqq 0 \qquad (1\text{-}1)$$

where A is a linear, in general unbounded, operator with domain D(A) and range R(A) both contained in a real Hilbert space H and f is a (nonlinear) function defined on H into H. It is well known that some semi-linear systems of differential equations, both ordinary and partial, can be reduced to the form (1-1) and in such cases A may be considered as an extension of a linear differential operator. In order to examine the stability of solutions to (1-1), it is only necessary to characterize their properties without actually constructing the solutions. This is done be considering the properties of a nonlinear semi-group because if the operator $A + f(\cdot)$ generates a nonlinear semi-group $\{T_t;\ t \geqq 0\}$ (see definition 2.1) then a solution to (1-1) starting at $t = 0$ from any element $x_0 \varepsilon D(A)$ is given by $x(t;\ x_0) = T_t x_0$ for all $t \geqq 0$ with $x(0;\ x_0) = x_0$. Thus the existence of a solution to (1-1) is ensured and the stability property can be determined from the family of nonlinear operators $\{T_t;\ t \geqq 0\}$. The object of this paper is to impose conditions on the operators A and f such that the operator $A_1 \equiv A+f(\cdot)$ generates a nonlinear contraction or negative contraction semi-group in H or in an equivalent Hilbert space of H (see definition 2.4) from which the existence, uniqueness and stability or asymptotic stability of solutions to (1-1) are insured.

The following definitions specifies what we mean by a solution, an equilibrium solution and the stability of an unperturbed solution.

<u>Definition 1.1</u>. By a solution $x(t)$ of (1-1) with initial conditions $x(0) = x\varepsilon D(A)$ in a Hilbert space H, we mean the following:

(a) $x(t)$ is uniformly Lipschitz continuous in t for each $t \geqq 0$ with $x(0) = x$;

(b) $x(t)\varepsilon D(A)$ for each $t \geqq 0$ and $Ax(t) + f(x(t))$ is weakly continuous in t;

(c) the weak derivative of $x(t)$ exists for all $t \geqq 0$ and equals $Ax(t) + f(x(t))$;

(d) the strong derivative $\frac{dx(t)}{dt}$ $(=Ax(t) + f(x(t))$ exists and is strongly continuous except at a countable number of values t.

The above definition of a solution $x(t)$ is in the sense of a weak solution since $x(t)$ satisfies (1-1) in the weak topology of H. However, by the condition (d), $x(t)$ is an almost everywhere strong solution in the sense that $x(t)$ satisfies (1-1) for almost all values of $t \geqq 0$ in the strong topology of H.

<u>Definition 1.2</u>. An equilibrium solution of (1-1) is an element x_e in D(A) satisfying (1-1) (in the weak topology) such that for any solution $x(t)$ of (1-1) with $x(0) = x_e$

$$||x(t)-x_e|| = 0 \qquad \text{for all } t \geqq 0.$$

It can be shown that if $x(t)$ is a solution of (1-1) then it is an equilibrium solution if and only if $Ax(t) + f(x(t)) = 0$ for all $t \geqq 0$ (cf [9]).

<u>Definition 1.3</u>. An equilibrium solution (or any unperturbed solution) x_e of (1-1) is said to be stable (with respect to initial perturbations) if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$||x-x_e|| < \delta \quad \text{implies} \quad ||x(t)-x_e|| < \varepsilon \quad \text{for all } t \geq 0;$$

x_e is said to be asymptotically stable if

(i) it is stable; and

(ii) $\lim_{t\to\infty} ||x(t)-x_e|| = 0$

where $x(t)$ is any solution of (1-1) with $x(0) = x \in D(A)$. If there exist positive constants M and β such

$$\text{(ii)}' \quad ||x(t)-x_e|| \leq Me^{-\beta t} \, ||x-x_e|| \qquad \text{for all } t \geq 0$$

then x_e is called exponentially asymptotically stable.

The existence problem for the case of a general nonlinear equation of evolution

$$\frac{dx(t)}{dt} = A_1 x(t), \tag{1-2}$$

where A_1 is a general nonlinear operator with $D(A_1)$ and $R(A_1)$ both in H, has been investigated by Kōmura [6], Kato [5] and by Browder [2]; and the stability problem of the same type of equation has been studied by the author in a separate paper [9] which has a close connection with the present work. Because of the implication of a general nonlinear operator differential equation, the application of its results to partial or ordinary differential equation needs additional justification. However, in the case of semi-linear equation of the form (1-1), criteria on the operator A and on the function f are directly applicable to certain partial or ordinary differential equations. Examples of application to partial differential equations are given in the last section in order to illustrate certain steps in applying the results developed for operator differential equation. Further applications to nonlinear partial differential equations will be discussed in a separate presentation.

2. General Background

In this section, we shall introduce some basic definitions and state some theorems from [9] which are fundamental in the development of our results.

Definition 2.1. Let H be a Hilbert space. The family of operators $\{T_t;\ t \geq 0\}$ is called a nonlinear semi-group on H if and only if the following conditions hold:

(i) for any fixed $t \geq 0$, T_t is a continuous (nonlinear) operator defined on H into H;

(ii) for any fixed $x \in H$, $T_t x$ is strongly continuous in t;

(iii) $T_s T_t = T_{s+t}$ for s, $t \geq 0$, and $T_o = I$ (the identity operator);

(iv) $||T_t x - T_t y|| \leq M||x-y||$ $(M > 0)$ $x,y, \in H$ and $t \geq 0$.

If (iv) is replaced by

(iv)' $||T_t x - T_t y|| \leq Me^{-\beta t}||x-y||$ $(\beta > 0)$ $x,y \in H$ and $t \geq 0$

then $\{T_t;\ t \geq 0\}$ is called a (nonlinear) negative semi-group; if $M \leq 1$ then it is called a (nonlinear) contraction or negative contraction semi-group according to (iv) or (iv)' respectively. The number β satisfying (iv)' is called a contractive constant of $\{T_t;\ t \geq 0\}$. For a subset D of H, the family $\{T_t;\ t \geq 0\}$ is said to be a nonlinear contraction (resp., negative contraction) semi-group on D if the properties (i)-(iv) (resp.,(i)-(iv)') are satisfied with $(M \leq 1)$ for all $x,y \in D$.

Definition 2.2. The infinitesimal generator A_1 of the nonlinear semi-group $\{T_t;\ t \geq 0\}$ is defined by

$$A_1 x = \underset{h \downarrow 0}{\text{w-lim}} \frac{T_h x - x}{h}$$

for all $x \in H$ such that the limit on the right-side exists in the sense of weak convergence.

Definition 2.3. An operator (nonlinear) A_1 with domain $D(A_1)$ and range $R(A_1)$ both contained in a real Hilbert space is said to be dissipative if

$$(A_1x - A_1y, x-y) \leqq 0 \qquad \text{for } x, y \in D(A_1); \qquad (2\text{-}1)$$

and A_1 is called strictly dissipative if there exists a real number $\beta > 0$ such that

$$(A_1x - A_1y, x-y) \leqq -\beta||x-y||^2 \qquad x,y \in D(A_1). \quad (2\text{-}2)$$

The supremum of all numbers β satisfying (2-2) is called the dissipative constant of A_1.

It follows from the above definition that A_1 is dissipative if and only if $-A_1$ is monotone and A_1 is strictly dissipative if and only if there exists a real number $\beta > 0$ such that $-(A_1 + \beta I)$ is monotone (cf [8]). Note that definition 2.3 coincides with the usual definition of dissipativity when A_1 is a linear operator (cf. [7]).

Definition 2.4. Two inner products $(\cdot \, , \cdot)$ and $(\cdot \, , \cdot)_1$ defined on the same vector space H are said to be equivalent if and only if the norms $||\cdot||$ and $||\cdot||_1$ induced by $(\cdot \, , \cdot)$ and $(\cdot \, , \cdot)_1$ respectively are equivalent, that is, there exist constants δ, γ with $0 < \delta \leqq \gamma < \infty$ such that

$$\delta||x|| \leqq ||x||_1 \leqq \gamma||x|| \qquad \text{for all } x \in H. \quad (2\text{-}3)$$

The Hilbert space H_1 equipped with the inner product $(\cdot \, , \cdot)_1$ is said to be an equivalent Hilbert space of H and is denoted by $(H, (\cdot \, , \cdot)_1)$ or simply by H_1.

In order to show the results in the following sections we state some results from [9].

Theorem 2.1. Let A_1 be a nonlinear operator with domain $D(A_1)$ and range $R(A_1)$ both contained in a Hilbert space H such that $R(I-A_1)=H$.

Then A_1 is the infinitesimal generator of a nonlinear contraction semi-group on $D(A_1)$ if and only if A_1 is dissipative; and A_1 is the infinitesimal generator of a nonlinear negative contraction semi-group if and only if A_1 is strictly dissipative.

Theorem 2.2. If A_1 is the infinitesimal generator of a nonlinear contraction semi-group (resp., negative contraction semi-group) in an equivalent Hilbert space on $D(A_1)$ then A_1 is the infinitesimal generator of a nonlinear semi-group (resp., negative semi-group), not necessarily contractive, on the same domain $D(A_1)$ in the original Hilbert space.

Remarks: (a) In Theorem 2.1, the condition $R(I-A_1) = H$ can be weakened by $R(I - \alpha A_1) = H$ for some $\alpha > 0$. (b) The nonlinear contraction semi-group $\{T_t;\ t \geqq 0\}$ generated by A_1 in the above theorem has the following additional property: For any $x \varepsilon D(A_1)$, the strong derivative $\frac{d(T_t x)}{dt} = A_1 T_t x$ exists and is strongly continuous except at a countable number of values t (cf. [5]). Thus for any $x \varepsilon D(A_1)$, $T_t x$ is a solution of (1-1) in the sense of definition 1.1.

It is seen from theorem 2.1 that if the operator $A + f(\cdot)$ is dissipative or strictly dissipative and $R(I - A - f(\cdot)) = H$ then $A + f(\cdot)$ is the infinitesimal generator of a contraction and negative contraction semi-group respectively. However, the requirement $R(I - A - f(\cdot)) = H$ by itself is not easy to verify since it is equivalent to the functional equation

$$x - Ax - f(x) = z \tag{2-4}$$

having a solution for every $z \varepsilon H$. In the following section, we shall impose conditions on A and f to insure the existence of a solution of (2-4). We consider first the case that A is the infinitesimal generator of a linear contraction (or negative contraction) semi-group of class C_o (cf. [11]), and then consider the more general case when A is an unbounded closed operator. Notice that the infinitesimal generator of a semi-group is always closed.

3. Existence and Stability of Solutions

In the proof of the main theorems in this section, we have used some results developed by Browder (cf. [1], [2]). It is noted that a Hilbert space is reflexive and uniformly convex and the definition of an accretive operator defined in [2] coincides with a monotone operator when the underlying space is a Hilbert space.

Definition 3.1. Let $x(t)$ be a solution to (1-1) with $x(0) = x$. A subset D of H is said to be a stability region of the equilibrium solution (or any unperturbed solution) x_e if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$x \in D \text{ and } ||x-x_e|| < \delta \quad \text{imply} \quad ||x(t)-x_e|| < \varepsilon \quad \text{for all } t \geqq 0.$$

Theorem 3.1. Let A be the infinitesimal generator of a (linear) contraction semi-group of class C_o. Assume that f satisfies: (i) f is defined on all of H into H such that it is continuous from H in the strong topology to the weak topology, and is bounded on every bounded subset of H. (ii) for any $x, y \in H$, $(f(x)-f(y), x-y) \leqq 0$. Then (a) for any $x \in D(A)$, there exists a unique solution of (1-1) (in the sense of definition 1.1) with $T_o x = x$; (b) any equilibrium solution x_e (or any unperturbed solution such as periodic solution), if it exists, is stable; (c) a stability region of x_e is D(A) which can be extended to H.

Proof. Let $A_1 = A + f(\cdot)$ with $D(A_1) = D(A)$. Since an infinitesimal generator of a contraction semi-group of class C_o is densely defined, dissipative and $R(I-A) = H$ (cf. [7] or [11]), it follows by the dissipativity of A and by the assumption (ii) on f that

$$(A_1x-A_1y,\ x-y) = (Ax-Ay,\ x-y) + (f(x)-f(y),\ x-y) \leqq 0 \quad \text{for all } x,y \in D(A_1)$$

which shows that A_1 is dissipative. To show that $R(I-A_1) = H$, we apply a theorem (theorem 5) in [2]. Note that the operator $-A$ is monotone and the range of $-A + I$ is all of H with $D(-A) = D(A)$ dense in H. Thus the operator $G = -A$ is accretive (or monotone). Let $G_o = I-f(\cdot)$, then from assumption (i) G_o is defined on all of H and is continuous from H in the strong topology to the weak topology (i.e., G_o is demicontinuous on H) and maps bounded subsets of H into bounded subsets of H. G_o is monotone, for

$$(G_ox-G_oy,\ x-y) = (x-y,\ x-y) - (f(x)-f(y),\ x-y) \geqq ||x-y||^2 \quad x,y \in H$$

where we have used assumption (ii). Moreover, by letting $y=0$ in (ii) gives

$$(f(x),x) \leqq (f(0),x) \leqq ||f(0)||\ ||x|| \qquad \text{for all } x \in H \tag{3-1}$$

It follows by the dissipativity of A and by (3-1) that

$$||-Ax+G_ox|| \geqq (-Ax+G_ox,x)/||x|| \geqq (G_ox,x)/||x|| = ((x,x)-(f(x),x))/||x||$$
$$\geqq ||x|| - ||f(0)|| \qquad \text{for all } x \in D(A) \quad (x \neq 0).$$

Thus $||Gx + G_ox|| \to +\infty$ as $||x|| \to \infty$. Hence all the hypotheses in theorem 5 of [2] are satisfied. It follows that $R(I-A_1) = R(G+G_o) = H$. This later condition and the dissipativity of A_1 imply that A_1 is the infinitesimal generator of a nonlinear contraction semi-group $\{T_t;\ t \geqq 0\}$ on $D(A)$ by applying theorem 2.1. Therefore, for any $x \in D(A)$, $T_tx \in D(A)$ and is the unique solution of 1-1 with $T_ox=x$. Since

$$||T_tx-T_ty|| \leqq ||x-y|| \qquad \text{for all } t \geqq 0 \qquad x,y \in D(A)$$

it follows that by taking y as the equilibrium solution x_e or any unperturbed solution such as periodic solution, if it exists, then it is stable. Note that $T_t x_e = x_e$. The above inequality holds for any $x, y \in D(A)$ which implies that a stability region is $D(A)$ and thus it can be extended to the whole space H since $D(A)$ is dense in H (cf. [9]). Therefore, the theorem is proved.

Theorem 3.2. Let A be the infinitesimal generator of a (linear) negative contraction semi-group of class C_0 with contractive constant β. Assume that f satisfies the condition (i) in theorem 3.1 and that

$$(f(x) - f(y), x-y) \leqq k||x-y||^2 \text{ with } k < \beta \quad \text{for all } x,y \in H. \tag{3-2}$$

Then all the results in theorem 3.1 hold. Moreover, if an equilibrium solution exists (or any unperturbed solution), it is exponentially asymptotically stable.

Proof. Let $A_1 = A + f(\cdot)$. Since A is the infinitesimal generator of a negative contraction semi-group, it is densely defined, dissipative and $R(I-A) = H$. Applying theorem 2.1 for the linear case, A is strictly dissipative with dissipative constant β, that is

$$(Ax,x) \leqq -\beta||x||^2 \qquad \text{for all } x \in D(A).$$

Thus the operator A_1 is strictly dissipative with dissipative constant β-k since by the hypothesis (3-2)

$$(A_1x-A_1y, x-y) = (Ax-Ay, x-y) + (f(x)-f(y), x-y) \leqq -(\beta-k)||x-y||^2$$

for all $x,y \in D(A_1)$. To show that $R(I-A_1) = H$, we prove $R(I-\alpha A_1) = H$ for some $\alpha > 0$, since the monotonicity of $-A$ implies that $(I-\alpha A)^{-1}$ exists for every $\alpha > 0$, and if $R(I-\alpha A) = H$ for some $\alpha > 0$ then $R(I-A) = H$ (cf. [5]). The reason for doing this is that if the same argument as in the proof of theorem 3.1 is used it will lead to the unnecessary requirement $k \leqq 1$. Let $I - \alpha A_1 = -\alpha A + (I - \alpha f(\cdot)) = G + G_o$ where $G = -\alpha A$, and $G_o = I - \alpha f(\cdot)$. Since A is the infinitesimal generator of a semi-group, $\alpha \in \rho(A)$ (the resolvent set of A) for all $\alpha > 0$ (cf. [11]) which implies that $R(I+G) = R(I - \alpha A) = H$. The mapping $G_o = I - \alpha f(\cdot)$ is monotone for $\alpha \leqq k^{-1}$ since by the assumption (ii)

$$(G_o x - G_o y, x-y) = (x-y, x-y) - \alpha(f(x)-f(y), x-y) \geqq (1-\alpha k)||x-y||^2 \geqq 0.$$

It is obvious by the assumption (i) that G_o is continuous on H in the strong topology to the weak topology and is bounded on every bounded subset of H. Finally, the relation $||Gx+G_o x|| \to \infty$ as $||x|| \to \infty$ is also satisfied. This is due to the fact that the dissipativity of αA and the relation (3-1) imply that

$$||Gx+G_o x|| \geqq (-\alpha Ax + G_o x, x)/||x|| \geqq ((x,x) - \alpha(f(x),x))/||x|| \geqq ||x|| - \alpha||f(0)||$$

where $\alpha > 0$ is a fixed number. Hence by choosing $\alpha \leqq k^{-1}$, all the hypotheses in theorem 5 of [2] are satisfied and the result $R(I - \alpha A) = R(G+G_o) = H$ follows. It should be noted that $k > 0$ so that $0 < \alpha \leqq k^{-1}$ exists. (if $k \leqq 0$, then G_o is monotone by taking, for instance, $\alpha = 1$ and the other conditions remain unchanged). By theorem 2.1, A_1 is the infinitesimal generator of a nonlinear negative contraction semi-group $\{T_t; t \geqq 0\}$ on D(A) with the contractive constant β-k. Therefore the results in the theorem follow directly from the negative contraction property of the semi-group $\{T_t; t \geqq 0\}$.

Remark. If A is the infinitesimal generator of a contraction semi-group instead of a negative contraction semi-group, any unperturbed solution is still asymptotically stable provided that the constant k appearing in the condition (3-2) is negative, since in this case, we may take $\beta=0$ and the operator $A_1 = A + f(\cdot)$ remains strictly dissipative with dissipative constant -k. The proof of $R(I-A) = H$ remains the same.

<u>Corollary 1.</u> Under the hypotheses of theorem 3.1 (theorem 3.2) and in addition, if $f(0) = 0$, then the null solution is stable (asymptotically stable) with the stability region the whole space H.

<u>Proof.</u> If $f(0) = 0$ then $x(t) \equiv 0$ is an equilibrium solution (called the null solution) of (1-1). Hence by theorem 3.1 (resp., theorem 3.2), the null solution is stable (resp., asymptotically stable) with the stability region extended to the whole space H.

<u>Corollary 2.</u> Let A be the infinitesimal generator of a (linear) negative contraction semi-group of class C_o with contractive constant β, and let f be Lipschitz continuous on H with Lipschitz constant $k < \beta$, that is

$$||f(x)-f(y)|| \leqq k||x-y|| \quad \text{for all } x,y \in H. \tag{3-3}$$

Then for any $x \in D(A)$ there exists a unique solution $T_t x$ to (1-1) with $T_o x=x$ such that any equilibrium solution x_e to (1-1) is asymptotically stable. In particular, if $f(0) = 0$ the null solution is asymptotically stable. Moreover, a stability region is $D(A)$ which can be extended to the whole space H.

<u>Proof.</u> By the Lipschitz continuity of f on H, it follows that condition (i) in theorem 3.1 is satisfied. This is due to the fact that strong continuity implies weak continuity, and by (3-3) with x_o a fixed element in H

$$||f(x)|| \leqq ||f(x_o)|| +k\ ||x|| +k\ ||x_o||$$

which is bounded whenever $||x||$ is bounded. Moreover, by (3-3)

$$(f(x) - f(y),\ x-y) \leqq ||f(x)-f(y)||\ ||x-y|| \leqq k||x-y||^2$$

and so condition (3-2) in theorem 3.2 is satisfied. Hence, by theorem 3.2 the existence and the uniqueness of a solution as well as the stability property of an equilibrium solution are proved. In particular, if $f(0)=0$ then corollary 1 implies that the null solution is asymptotically stable.

It is obvious that under the hypotheses of theorem 3.2 and in addition if an equilibrium solution x_e exists then it is unique since if y_e is another equilibrium solution, the negative contraction property of any two solutions to (1-1) implies that

$$||x_e - y_e|| \leq e^{-(\beta-k)t}||x_e - y_e|| \qquad \text{for all } t \geq 0$$

which is impossible unless $x_e = y_e$. Note that $T_t x_e = x_e$ and $T_t y_e = y_e$ for all $t \geq 0$. The following theorem gives weaker conditions on A_o and on f for the uniqueness of an equilibrium solution.

Theorem 3.3. Let the linear operator A appearing in (1-1) be such that $0 \in D(A)$ and that for some finite number β(i.e., $|\beta|<\infty$),

$$(Ax,x) \leq \beta(x,x) \qquad \text{for all } x \in D(A).$$

Let f be defined on $D(A)$ to H such that $f(0)=0$ and such that for some finite number k (i.e., $|k| < \infty$)

$$(f(x),x) \leq k||x||^2 \qquad \text{for all } x \in D(A).$$

If $\beta > k$ then the null solution of (1-1) is the only equilibrium solution.

Proof. It is obvious that the zero vector is an equilibrium solution of (1-1). Let x_e be any other equilibrium solution, then $x_e \in D(A)$ and by the definition of an equilibrium solution, $Ax_e + f(x_e) = 0$. It follows that

$$0 = (Ax_e + f(x_e),x_e) = (Ax_e,x_e) + (f(x_e),x_e) \leq -(\beta-k)||x_e||^2$$

which implies that $x_e = 0$ since by hypothesis $\beta-k > 0$. Hence the uniqueness of the equilibrium solution is proved.

Most of the theorems developed in this section up to now assumed that the linear part A of (1-1) is the infinitesimal generator of a contraction semi-group of class C_o. A necessary and sufficient condition for A having this property is that A is dissipative, $D(A)$ is dense in H

and $R(I-A) = H$ (cf. [7]). Again the requirement $R(I-A) = H$ means the existence of a solution of the functional equation

$$x - Ax = z$$

for every $z \in H$ which by itself needs further justification. However in case A is a self-ajoint operator which occurs often in physical applications, this requirement can be eliminated in these theorems. In order to show this, we apply a theorem from [1] due to Browder by considering a densely defined closed operator and then take a self-adjoint operator as a special case.

Theorem 3.4. Let A be a densely defined closed operator from H into H. Suppose that: (i) A is strictly dissipative with dissipative constant β, (ii) A^* is the closure of its restriction to $D(A) \cap D(A^*)$ where A^* is the adjoint operator of A, (iii) f satisfies the conditions (i) and (3-2) in theorem 3.2. Then all the results in theorem 3.2 hold.

Proof. Let $A_1 = A + f(\cdot)$, then A_1 is strictly dissipative, since by hypothesis

$$(A_1x-A_1y,x-y) = (Ax-Ay,x-y) + (f(x)-f(y),x-y) \leqq -(\beta-k)||x-y||^2$$

for all $x,y \in D(A) = D(A_1)$. To show that $R(I-A_1) = H$, let $T=I-A_1=-A+(I-f(\cdot))$, then $D(T)=D(A)$ is densely defined. Since $-A$ is densely defined, A^* exists and is closed, and by the assumption (ii) $-A^*$ is the closure of its restriction to $D(-A) \cap D(-A^*)$. By (iii) the operator $G=I-f(\cdot)$ is continuous from all of H to H in the strong topology to the weak topology which implies its hemi-continuity from H to H with $D(G) = H$. The boundedness of G on bounded subsets of H also follows from (iii). Moreover

$$(Tx-Ty,x-y) = (x-y,x-y) - (A_1x-A_1y,x-y) \geqq (1+\beta-k)||x-y||^2 \quad x,y \in D(T)$$

so that T is monotone. In particular by letting $y=0$ ($0 \in D(A)=D(T)$) in the above inequality and since $T\cdot 0 = 0-A_1\cdot 0 = -f(0)$, it follows that

$$(Tx,x) \geqq (1+\beta-k)||x||^2 - (f(0),x) \geqq ((1+\beta-k)||x||-||f(0)||)||x||$$

for all $x \in D(T)$,

and since $\beta - k > 0$ the real valued function $c(||x||)$ defined by

$$c(||x||) = (1+\beta-k)||x||-||f(0)||$$

has the property that $c(||x||) \to \infty$ as $||x|| \to \infty$. Hence all the conditions in theorem 1 of [1] are satisfied if we take, for instance, the completely continuous mapping $C=0$ (the zero operator which maps all $x \in H$ into the 0 vector in H). Therefore $R(I-A_1) = R(T) = H$. By applying theorem 2.1, A_1 is the infinitesimal generator of a non-linear negative contraction semi-group on $D(A_1) = D(A)$ with the contractive constant $\beta-k$. Thus, the stated results in the thoerem follow directly from the negative contraction semi-group property as in the proof of theorem 3.1.

Remark. The above theorem can also be proved with $\beta=k=0$, in which case the equilibrium solution is stable with a stability region $D(A)$. The proof is exactly the same by letting $\beta=k=0$.

Since an unbounded self-adjoint operator A is a densely defined closed operator having the property that $D(A) = D(A^*)$ (in fact $A = A^*$) we have, with a stronger assumption on the function f, the following result which is stated as a theorem becasue of its usefulness in applications.

Theorem 3.5. Let A be an unbounded self-adjoint operator from H to H and assume that it is strictly dissipative with dissipative constant β. Let f be Lipschitz continuous on H with Lipschitz constant $k < \beta$, that is

$$||f(x)-f(y)|| \leqq k||x-y|| \qquad \text{for all } x,y \in H.$$

Then all the results in theorem 3.2 hold.

Proof. The self-adjointness of A implies that A is a densely defined closed operator and $D(A^*) = D(A)$ (in fact, $A=A^*$). Thus condition (ii) in theorem 3.4 is satisfied. By the Lipschitz continuity of f, f is continuous in the strong topology and is bounded on every bounded subset of H. This assumption (Lipschitz continuity) also implies that

$$(f(x)-f(y),x-y) \leqq ||f(x)-f(y)||\ ||x-y|| \leqq k||x-y||^2 \quad \text{for all } x,y \in H.$$

Hence, all the conditions in theorem 3.4 are satisfied, and the results follow by applying that theorem.

Remark. The Lipschitz continuity of f in the theorem can be weakened by using the conditions (i) and (3-2) in theorem 3.2.

It is easily seen from theorems 2.1 and 2.2 that stability and asymptotic stability are invariant if the inner product $(\cdot,\cdot)$ of H is replaced by an equivalent inner product $(\cdot,\cdot)_1$ with respect to which $A_1 = A + f(\cdot)$ is dissipative. Because of its usefulness in applications (for instance, a non-self-adjoint operator in a Hilbert space $(H,(\cdot,\cdot))$ can sometimes be made self-adjoint in $(H,(\cdot,\cdot)_1)$ where $(\cdot,\cdot)_1$ is an equivalent inner product) we show the following theorem.

Theorem 3.6. Let A be a densely defined linear operator from $H=(H,(\cdot,\cdot))$ into H, and let f satisfy the condition (i) in theorem 3.1. If there exists an equivalent inner product $(\cdot,\cdot)_1$ such that A is a self-adjoint operator in $H_1 = (H,(\cdot,\cdot)_1)$ satisfying

$$(Ax,x)_1 \leqq -\beta||x||_1^2 \qquad x \in D(A)$$

and if

$$(f(x)-f(y),x-y)_1 \leqq k||x-y||_1^2 \quad \text{with } k < \beta, \quad x,y \in H.$$

Then, all the results stated in theorem 3.4 are valid.

Proof. Consider A as an operator from the space $H_1 = (H,(\cdot,\cdot)_1)$ into H_1. Since A is self-adjoint in the space H_1, it is a densely defined closed operator and $D(A) = D(A^*)$. The continuity and the boundedness of f with respect to the $||\cdot||$-norm topology implies the same property of f with respect to the $||\cdot||_1$-norm topology since these two norms are equivalent. By assumption, A is strictly dissipative and the condition (iii) in theorem 3.4 is satisfied with respect to $(\cdot,\cdot)_1$. Hence all the hypotheses in theorem 3.4 are satisfied by considering H_1 as the underlying space which implies that the operator $A_1 = A + f(\cdot)$ is the infinitesimal generator of a nonlinear negative contraction semi-group $\{T_t;\ t \geqq 0\}$ on $D(A)$ with contractive constant $\beta - k$ in the space H_1. By theorem 2.2, A is the infinitesimal generator of a nonlinear negative semi-group $\{T_t;\ t \geqq 0\}$ on $D(A)$, not necessarily contractive, in the original space H. Therefore all the results in theorem 3.4 hold by the semi-group properties.

4. Applications to Partial Differential Equations

In this section, we shall give some applications of the results obtained in the previous section to a class of linear and semi-linear partial differential equations which can serve as an illustration of some steps in applying the theorems developed for operator differential equations. For simplicity, we limit our discussion to second order differential equations in an n-dimensional Euclidian space R^n and consider the Hilbert space $L^2(\Omega)$ as the underlying space. In the following, the first simple example of a linear differential equations gives a fairly detailed description of the application from which some more general equations can easily be obtained. Criteria for the existence and stability of solutions in terms of the coefficients of a given partial differential operator are stated as theorems which are concrete results of the application from the abstract operator differential equation.

Example 4.1. Consider the linear partial differential equation

$$\frac{\partial u}{\partial t} = a(\chi)\frac{\partial^2 u}{\partial \chi^2} + b(\chi)\frac{\partial u}{\partial \chi} + c(\chi)u \qquad \chi \varepsilon (0,1) \qquad (4\text{-}1)$$

with the boundary conditions

$$u(t,0) = u(t,1) = 0 \qquad (t \geqq 0). \qquad (4\text{-}2)$$

Assume that the coefficient $a(\chi)$ is positive on $[0,1]$ and that $a(\chi)$, $b(\chi)$, $c(\chi)$ are all infinitely differentiable functions in an open interval I_o containing $[0,1]$. Then the linear operator

$$L = a(\chi)\frac{\partial^2}{\partial \chi^2} + b(\chi)\frac{\partial}{\partial \chi} + c(\chi)$$

is an elliptic partial differential operator (cf. [4]). The formal adjoint operator of L is given as

$$L^*(\cdot) = \frac{\partial^2}{\partial \chi^2}(a(\chi)(\cdot)) - \frac{\partial}{\partial \chi}(b(\chi)(\cdot)) + c(\chi)(\cdot)$$

which is also an elliptic partial differential operator. It is easily shown by a simple calculation that equation (4-1) can be reduced to the form

$$\frac{\partial u}{\partial t} = \frac{1}{q(\chi)}\ \frac{\partial}{\partial \chi}\left(p(\chi)\frac{\partial}{\partial \chi}\right) + c(\chi)u \qquad (4\text{-}3)$$

where

$$q(\chi) = (a(\chi))^{-1} e^{\int_{\chi_o}^{\chi}(b(\xi)/a(\xi))d\xi} \qquad (\chi_o \ \varepsilon\ [0,\chi] \text{ fixed})$$

$$p(\chi) = a(\chi)\,q(\chi). \qquad (4\text{-}4)$$

Let us seek a solution in the real Hilbert space $L^2(0,1)$ in which the inner product between any pair of elements u, v ε $L^2(0,1)$ is defined by

$$(u, v) = \int_o^1 u(\chi)\ v(\chi)\ d\chi\ . \qquad (4\text{-}5)$$

Define the operator T in $L^2(0,1)$ as the restriction of L on $C^\infty(0,1)$ and $\hat{T}$ the restriction of L* on $C^\infty(0,1)$ by

$$D(T) = D(\hat{T}) = \{u \ \varepsilon\ C^\infty([0,1]); u(0) = u(1) = 0\}$$

$$Tu = Lu, \quad \hat{T}u = L^*u \qquad u\ \varepsilon\ D(T).$$

Let A and $\hat{A}$ denote the closure of T and $\hat{T}$ respectively (T and $\hat{T}$ are closable). Then $D(A)$ is dense in $L^2(0,1)$ since $D(A) \supset D(T) \supset C_o^\infty(0,1)$ which is dense in $L^2(0,1)$. Thus A^* and $(\hat{A})^*$ both exist. In general, T is not self-adjoint with respect to the inner product defined in (4-5). However, by defining the scalar functional $(u,v)_1$ by

$$(u,v)_1 = (u,qv) = \int_o^1 u(\chi)\; q(\chi)\; v(\chi)\; d\chi \tag{4-6}$$

where the function $q(\chi)$ is the known function given in (4-4) it is easily seen that $(\cdot,\cdot)_1$ possesses all the properties of an inner product. Since $(u,u)_1 = (u,qu) = \int_o^1 qu^2 d\chi$, it follows that

$$\left(\min_{0\leq\chi\leq 1} q(\chi)\right) ||u||^2 \leq ||u||_1^2 \leq \left(\max_{0\leq\chi\leq 1} q(\chi)\right) ||u||^2$$

which implies that $(\cdot,\cdot)_1$ and $(\cdot,\cdot)$ are equivalent. Notice that $q(\chi)>0$ and is continuous over the closed interval $[0,1]$ so that it actually attains its maximum and minimum values bounded away from zero and ∞. Moreover, for any u, $v \in D(T)$, on integrating by parts and taking notice that the boundary conditions are satisfied for any $u \in D(T)$ we have

$$(u,Tv)_1 = (u,qTv) = \int_o^1 uq[q^{-1} \frac{\partial}{\partial\chi}(p\, \frac{\partial v}{\partial\chi}) + cv]d\chi$$

$$= \int_o^1 [v\, \frac{\partial}{\partial\chi}\, (p\, \frac{\partial u}{\partial\chi}) + c\; q\; u\; v]d\chi = (Tu,v)_1 \tag{4-7}$$

which shows that $T=\hat{T}$. It follows that (cf. Dunford and Schwartz [4] p. 1740) $A=(\hat{A})^*=A^*$ which shows that A is self-adjoint in the equivalent Hilbert space $L_1^2(0,1)$ equipped with the inner product $(\cdot,\cdot)_1$. Moreover, the above equality implies that for any $u \in D(T)$

$$(u,Tu)_1 = - \int_o^1 [p(\frac{\partial u}{\partial\chi})^2 - c\; q\; u^2]d\chi = - \int_o^1 [a\; q(\frac{\partial u}{\partial\chi})^2 - c\; q\; u^2]d\chi.$$

On setting $u_1 = q^{1/2}\, u$ then $||u_1||=||u||_1$ and by an elementary calculation we have

$$aq(\frac{\partial u}{\partial\chi})^2 = a(\frac{\partial u_1}{\partial\chi})^2 - \frac{1}{2}\, (b-a')\, \frac{\partial u_1^2}{\partial\chi} + \frac{1}{4}\, \frac{(b-a')^2}{a}\, u_1^2 \tag{4-8}$$

where $a' \equiv \frac{d}{dx} a(x)$. Hence, integrating by parts and using the well known inequality

$$\int_0^1 \left(\frac{du}{d\chi}\right)^2 d\chi \geq \pi^2 \int_0^1 u^2 \, d\chi \tag{4-9}$$

which is valid for any $u(x)$ satisfying the condition (4-2), we have

$$(u,Tu)_1 = -\int_0^1 \left[a\left(\frac{\partial u_1}{\partial \chi}\right)^2 + \left(\frac{1}{2}(b'-a'') + \frac{1}{4}\frac{(b-a')^2}{a} - c\right) u_1^2\right] d\chi$$

$$\leq -\int_0^1 \left[\pi^2 a_{min} + \frac{1}{2}(b'-a'') + \frac{1}{4}\frac{(b-a')^2}{a} - c\right] u_1^2 \, d\chi \leq -\beta \| u \|_1^2$$

where

$$a_{min} = \min_{0 \leq \chi \leq 1} a(\chi)$$

$$\beta = \min_{0 \leq \chi \leq 1} \left[\pi^2 a_{min} + \frac{1}{2}(b'(\chi)-a''(\chi)) + \frac{1}{4}\frac{(b(\chi)-a'(\chi))^2}{a(\chi)} - c(\chi)\right]. \tag{4-10}$$

It follows that if $\beta=0$ or $\beta>0$ then T is dissipative or strictly dissipative, respectively, with respect to $(\cdot,\cdot)_1$. The dissipativity or strict dissipativity of T implies the dissipativity or strict dissipativity, respectively, of A. To see this, let $u \in D(A)$ then by the definition of the closure of a closable operator there exists a sequence $\{u_n\} \subset D(T)$ such that $u_n \to u$ and $\lim_{n\to\infty} Tu_n$ exists and equals Au (cf. [11]). Hence by the continuity of inner product, we have

$$(Au,u)_1 = \lim_{n\to\infty} (Tu_n,u_n)_1 \leq \lim_{n\to\infty} (-\beta \| u_n \|_1^2) = -\beta \| u \|_1^2$$

which shows the dissipativity and strict dissipativity of A. Therefore, by applying theorems 3.6 and 3.3 with $f \equiv 0$ we have the following results.

<u>Theorem 4.1</u>. Assume that the coefficients $a(\chi)$, $b(\chi)$ and $c(\chi)$ of 4-1 are infinitely differentiable over any open interval I_o containing $[0,1]$ and that $a(\chi)$ is positive on $[0,1]$. If the condition (4-10) is satisfied, then for any initial element $u_o(\chi) \in D(A)$ there exists a unique solution $u(t,\chi)$ in the sense of definition 1.1 with $u(0,\chi)=u_o(\chi)$. Moreover,

the null solution of (4-1) is stable if $\beta=0$ and is asymptotically stable if $\beta > 0$ and in the later case the null solution is the only equilibrium solution.

As an example of the above theorem, take $a(\chi) = \frac{1}{R}$, $b(\chi) = \frac{2}{\sqrt{R}}\chi$, $c(\chi) = (\chi^2 + \frac{2}{\sqrt{R}})$ where R is a positive constant to be determined, then

$$\beta = \min_{0\leq\chi\leq 1} \left[\frac{\pi^2}{R} + \frac{1}{\sqrt{R}} + \frac{1}{4} R \left(\frac{2}{\sqrt{R}}\chi\right)^2 - \left(\chi^2 + \frac{2}{\sqrt{R}}\right)\right] = \frac{\pi^2}{R} - \frac{1}{\sqrt{R}} .$$

Hence $\beta > 0$ if $0 < R < \pi^4$ which shows the same result as given in [3].

Remark. The solution $u(t,\chi)$ in theorem 4.1 is in fact a solution of (4-1) in the strong sense i.e., $\frac{du(t,\chi)}{dt} = Au(t,\chi)$ in the norm topology (cf. [7]). However, in the case of semi-linear equations, it is not certain that this is the case. Thus, we shall assume that any solution in the following discussion is in the sense of definition 1.1.

Example 4.2. Consider the partial differential equation

$$\frac{\partial u}{\partial t} = a(\chi)\frac{\partial^2 u}{\partial \chi^2} + b(\chi)\frac{\partial u}{\partial \chi} + c(\chi)u + f(\chi,u) \tag{4-11}$$

with the boundary conditions $u(t,0)=u(t,1)=0$ where $a(\chi)$, $b(\chi)$, $c(\chi)$ are the same as in theorem 4.1 and f is a nonlinear function defined on $L^2(0,1)$ to $L^2(0,1)$. According to theorem 3.6, if f is continuous on $L^2(0,1)$ and is bounded on bounded subsets of $L^2(0,1)$ such that

$$(f(\chi,u)-f(\chi,v),u-v)_1 \leq k_1||u-v||_1^2 \text{ with } k_1 < \beta, \quad u,v \in L^2(0,1)$$

where $(.,.)_1$ is the equivalent inner product defined in (4-6) and β is given by (4-10), then all the results in theorem 4.1 with respect to an equilibrium solution, if it exists, remain valid. In particular if $f(0)=0$, the null solution is exponentially asymptotically stable.

To illustrate the above statement take, for example, the function

$$f(\chi,u) = k(\chi)\frac{u^2}{\lambda^2+u^2} \qquad (\lambda^2 > 0).$$

where $k(\chi)$ is a bounded function on $[0,1]$. It is obvious that f is continuous on $L^2(0,1)$ (in the strong topology) and is bounded on $L^2(0,1)$. By the definition of $(.,.)_1$ in (4-6)

$$(f(\chi,u)-f(\chi,v),u-v)_1 = \int_o^1 k(\chi)\left(\frac{u^2}{\lambda^2+u^2} - \frac{v^2}{\lambda^2+v^2}\right)q(u-v)d\chi$$

$$= \lambda^2 \int_o^1 \frac{k(\chi)(u+v)}{(\lambda^2+u^2)(\lambda^2+v^2)} q(u-v)^2 \, d\chi \leqq$$

$$\leqq \lambda^2 \max_{0\leqq\chi\leqq 1} \left[\frac{|k(\chi)(u(\chi)+v(\chi))|}{(\lambda^2+u^2(\chi))(\lambda^2+v^2(\chi))}\right] ||u-v||_1^2 .$$

It is easily shown that for any real number u, v

$$\frac{|u+v|}{(\lambda^2+u^2)(\lambda^2+v^2)} < \frac{1}{|\lambda^3|}$$

which implies that

$$(f(\chi,u)-f(\chi,v),u-v)_1 < \left|\frac{k_m}{\lambda}\right| \; ||u-v||_1^2 .$$

where $k_m = \max_{0\leqq\chi\leqq 1} k(\chi)$ it follows that if $\left|\frac{k_m}{\lambda}\right| \leqq \beta$ then the existence and uniqueness of a solution for any initial element $u_o(\chi) \in D(A)$ are ensured. Moreover the null solution is exponentially asymptotically stable with stability region $D(A)$.

Example 4.3. Consider the second order linear differential equations of the form

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial \chi_i}\left(a_{ij}(\chi)\frac{\partial u}{\partial \chi_j}\right) + c(\chi)u \qquad \chi \in \Omega \qquad (4\text{-}12)$$

with the boundary conditions

$$u(t,\chi')=0 \qquad \chi' \in \partial\,\Omega \qquad t \geqq 0 \qquad (4\text{-}13)$$

where $\chi=(\chi_1,\chi_2,\dots,\chi_n)$, Ω is a bounded open subset of the Euclidean space R^n with boundary $\partial\Omega$ which is a smooth surface and no point in $\partial\Omega$ is interior to $\bar{\Omega}$, the closure of Ω. Assume that $a_{ij}(\chi) = a_{ji}(\chi)$ $(i,j=1,2,\dots,n)$ and together with $c(\chi)$ are infinitely differentiable real-valued functions in a domain Ω_o which contains $\bar{\Omega}$ and that there exists a positive constant α such that

$$\sum_{i,j=1}^{n} a_{ij}(\chi)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2 \qquad \chi \in \Omega_o, \ \xi \in R^n. \tag{4-14}$$

The operator

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial \chi_i} (a_{ij}(\chi) \frac{\partial}{\partial \chi_j}) + c(\chi)$$

is a strongly elliptic partial differential operator in Ω_o.

It is easily seen by definition that the operator L is formally self-adjoint i.e., L=L*. Let T be the operator in $L^2(\Omega)$ defined by

$$D(T) = \{u \in C^\infty(\bar{\Omega}); \ u(\chi') = 0, \ \chi' \in \partial \Omega\}$$

$$Tu = Lu \qquad\qquad u \in D(T),$$

and let A be the closure of T. By theorem 25 in [4] (p. 1743), A is self-adjoint. For any $u \in D(T)$, integration by parts yields

$$(u,Tu) = \int_\Omega uTud\chi = \int_\Omega [\sum_{i,j=1}^{n} u \frac{\partial}{\partial \chi_i}(a_{ij}(\chi) \frac{\partial u}{\partial \chi_j}) + c(\chi)u^2]d\chi$$

$$= - \int_\Omega [\sum_{i,j=1}^{n} a_{ij}(\chi) \frac{\partial u}{\partial \chi_i}\frac{\partial u}{\partial \chi_j} - c(\chi)u^2]d\chi$$

where $d\chi = d\chi_1 d\chi_2 \cdots d\chi_n$. By the assumption (4-14) and using the well known inequality [12]

$$\int_\Omega \sum_{i=1}^{n} (\frac{\partial u}{\partial \chi_i})^2 d\chi \geq \gamma \int_\Omega u^2 \, d\chi \tag{4-15}$$

where γ is a positive real number, we obtain

$$(u,Tu) \leq - \int_\Omega [\alpha \sum_{i=1}^{n} (\frac{\partial u}{\partial \chi_i})^2 - c(\chi)u^2]d\chi \leq -\int_\Omega (\alpha\gamma - c(\chi))u^2 d\chi$$

$$\leq -(\alpha\gamma - c_m)||u||^2 = -\beta||u||^2$$

where $c_m = \max_{\chi \in \bar{\Omega}} c(\chi)$ and $\beta = \alpha\gamma - c_m$. Hence, T is dissipative if $\beta=0$ and is strictly dissipative if $\beta>0$. The dissipativity and strict dissipativity of A follow from the dissipativity and strict dissipativity, respectively, of T as has been shown in example 4.1 since A is the closure of T. Therefore, A satisfies all the hypotheses in theorem 3.5. To summarize, we have:

Theorem 4.2. Assume that all the real-valued functions $a_{ij}(\chi)=a_{ji}(\chi)$ $(i,j=1,2,\ldots,n)$ and $c(\chi)$ in equation (4-12) are infinitely differentiable in a domain Ω_o containing $\bar{\Omega}$, the closure of Ω, where Ω is a bounded open set in R^n whose boundary $\partial\Omega$ is a smooth surface and no point of $\partial\Omega$ is interior to $\bar{\Omega}$. If the condition (4-14) is satisfied and if

$$\beta = \alpha\gamma - \max_{x\varepsilon\bar{\Omega}} c(\chi) \geqq 0 \qquad (4\text{-}16)$$

where α is given in (4-14) and γ is given in (4-15), then for any $u_o(\chi) \in D(A)$ there exists a unique solution $u(t,\chi)$ to (4-12) strongly continuous in t with respect to the $L^2(\Omega)$ norm with $u(0,\chi)=u_o(\chi)$. Moreover, the null solution is stable for $\beta=0$ and is asymptotically stable if $\beta > 0$ and in the later case the null solution is the only equilibrium solution. The stability region is $D(A)$ which can be extended to the whole space $L_2(\Omega)$.

It is seen from the above theorem that the major conditions imposed on the coefficients of the operator L are conditions (4-14) and (4-16). Notice that if $c(\chi)$ is a non-positive function, then (4-16) is automatically satisfied. As a special form of (4-12) we consider the equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial}{\partial\chi_i} \left(a_i(\chi) \frac{\partial u}{\partial\chi_i}\right) + c(\chi)\, u \qquad \chi \in \Omega \qquad (4\text{-}17)$$

with the boundary conditions (4-13). The following result is an immediate consequence of theorem 4.2.

Corollary. Assume that the real-valued functions $a_i(\chi)$ $(i=1,2,\ldots,n)$ and $c(\chi)$ in equation (4-17) are infinitely differentiable in a domain Ω_o

containing $\bar{\Omega}$ where Ω is a bounded open set in R^n whose boundary $\partial\Omega$ is sufficiently smooth. If, in addition, $a_i(\chi)$ is positive for each i and $c(\chi)$ is non-positive then all the results in theorem 4.2 hold.

Proof. Consider (4-17) as a special form of (4-12) with $a_{ij}(\chi) = a_i(\chi)$ for i=j and $a_{ij}(\chi)=0$ for $i\neq j$. Then the condition (4-14) is satisfied since by hypothesis $\alpha = \min\limits_{1\leq i\leq n} (\min\limits_{\chi\epsilon\bar{\Omega}} a_i(\chi))>0$ which implies

$$\sum_{i,j=1}^{n} a_{ij}(\chi)\ \xi_i\xi_j = \sum_{i=1}^{n} a_i(\chi)\xi_i^2 \geq \alpha \sum_{i=1}^{n} \xi_i^2.$$

The condition (4-16) follows from the non-positivity of $c(\chi)$. Hence the results follow by applying theorem 4.2.

As an example of the above theorem, consider the equation

$$\frac{du}{dt} = \Delta\ u-c^2u \qquad\qquad (c \text{ real})$$

where Δ is the Laplacian operator in $\Omega\subset R^3$ with $\partial\Omega$ sufficiently smooth. Then all the conditions in the above theorem are fulfilled since in this case $a_i(x) = 1$ for each i and $c(\chi) = -c^2$.

Just as one-dimensional space case, semi-linear equations of the form

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial\chi_i} (a_{ij}(\chi) \frac{\partial u}{\partial\chi_j}) + c(\chi)u + f(\chi,u) \qquad \chi\ \epsilon\ \Omega \qquad (4\text{-}18)$$

with the boundary conditions

$$u(t,\chi') = 0 \qquad\qquad \chi'\ \epsilon\ \partial\ \Omega \qquad\qquad (4\text{-}19)$$

can similarly be treated where f is a function on $L^2(\Omega)$ to $L^2(\Omega)$. For the sake of application, we state a theorem which is the consequence of theorem 3.5.

Theorem 4.3. Suppose that the semi-linear equation (4-18) with the boundary conditions (4-19) possesses the same linear part as given in theorem 4.2. If f satisfies the conditions (i) and (3-2) in theorems 3.1 and 3.2, respectively, where β is given by (4-16). Then (a) For any

$u_o(\chi) \in D(A)$ there exists a unique solution of (4-18) with $u(0,\chi)=u_o(\chi)$. (b) An equilibrium solution (or a periodic solution), if it exists, is stable if $k=\beta$; and is asymptotically stable if $k<\beta$. (c) A stability region of the equilibrium solution is D(A) which can be extended to the whole space $L^2(\Omega)$.

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